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# DIFFRACTION OF KELVIN WAVES AT THE OPEN END OF A PLANE-PARALLEL CHANNEL *

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The Wiener-Hopf method is used for solving the problem of surface wave diffraction at the end of a plane parallel channel in a tank rotating at constant velocity. Asymptotic and numerical analysis of the obtained solution is carried out.

**1. Statement of the problem.** Let us consider an infinite tank of finite depth $h$ rotating counterclockwise at angular velocity $\omega$. Two semi-infinite vertical walls are fixed to the tank bottom. The system of coordinates in which the equations of walls are $y = \pm a$, $x < 0$, where $2a$ is the channel width, is shown in Fig.1.

We consider in this tank a harmonic wave motion of the fluid surface whose elevation can be represented in the form $\xi(x, y) \exp(-i\sigma t)$, where $\sigma$ is the frequency of these oscillations. Let us consider the case of $\sigma > 2\omega$. In the theory of long surface waves /1/ function $\xi(x, y)$ is the solution of the wave equation

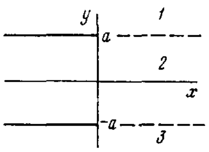
Fig.1

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \varkappa^2\right)\xi(x, y) = 0, \quad \varkappa^2 = \frac{\sigma^2 - 4\omega^2}{gh}$$

where $g$ is the acceleration of gravity.

Let the unit amplitude Kelvin wave

$$\xi_0(x, y) = \exp[i\eta\varkappa x - l\eta\varkappa(y - a)] \tag{1.1}$$
$$(l = 2\omega / \sigma, \ \eta = (1 - l^2)^{-1/2})$$

propagate from infinity in region $y > a$, $x < 0$ along the channel semi-infinite wall. Owing to the diffraction of wave (1.1) at the wall edge, waves which differ from $\xi_0$ are generated in the tank. We investigate these waves below.

We divide the tank in three regions, as shown in Fig.1. The total elevation amplitude in region 1 $(y > a)$ is defined by $\xi_0 + \xi_1$, where $\xi_0$ and $\xi_1$ are, respectively, the incident and diffracted wave amplitudes. In regions 2 $(|y| < a)$ and 3 $(y < -a)$ the elevation amplitudes are denoted by $\xi_2$ and $\xi_3$, respectively. For the unknown functions $\xi_j$ $(j = 1, 2, 3)$ we have the boundary value problem of finding the solution of equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \varkappa^2\right)\xi_j(x, y) = 0 \tag{1.2}$$

that would satisfy the boundary conditions at the channel walls and the conditions of continuity of the $y$-components of velocities and elevations along the extension lines of walls

$$v_0(x, a + 0) + v_1(x, a + 0) = 0 \tag{1.3}$$
$$v_2(x, |a - 0|) = 0, \ v_3(x, -a - 0) = 0 \quad (x < 0)$$
$$v_0(x, a + 0) + v_1(x, a + 0) = v_2(x, a - 0)$$
$$v_2(x, -a + 0) = v_3(x, -a - 0)$$
$$\xi_0(x, a + 0) + \xi_1(x, a + 0) = \xi_2(x, a - 0)$$
$$\xi_2(x, -a + 0) = \xi_3(x, -a - 0) \ (x > 0)$$

where $v_j(x, y)$ is the fluid velocity component parallel to the $y$-axis, which is related to $\xi_j(x, y)$ by the formula

$$v_j(x, y) = -\frac{\sigma}{\varkappa^2 h}\left(l\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\xi_j(x, y) \tag{1.4}$$

Finally, the diffracted waves must satisfy in the neighborhood of point $x = 0$, $y = \pm a$ the "condition at the edge"

$$\xi_j \sim r^{1/2} \quad (j = 1, 2, 3), \quad r = \sqrt{x^2 + (y \pm a)^2} \tag{1.5}$$

and the condition of radiation at infinity, viz. the solution must only contain divergent waves.

It can be shown that in the class of bounded functions the problem (1.1)—(1.5) has a unique solution.

**2. The system of paired integral equations and its solution.** We shall

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solve problem (1.1)—(1.5) using the Wiener—Hopf method /2/. For this we assume that the wave number $\varkappa$ has a small positive imaginary part, i.e. $\varkappa = \varkappa_0 + i\varepsilon$, and in the final results make $\varepsilon$ approach zero. The introduction in $\varkappa$ of the imaginary component conforms with the assumption that the fluid absorbs the energy of wave motions.

We introduce the unknown function $Z_1(\alpha)$, $Z_2(\alpha)$, $Z_2'(\alpha)$, $Z_3(\alpha)$, $A(\alpha)$ and $B(\alpha)$ of the complex variable $\alpha$ by formulas

$$\xi_{1,3}(x, y) = \int_{-\infty}^{+\infty} Z_{1,3}(\alpha) \exp[i\alpha x + i\gamma(\pm y - a)]\, d\alpha \tag{2.1}$$

$$\xi_2(x, y) = \int_{-\infty}^{+\infty} [A(\alpha)\sin\gamma(y + a) + B(\alpha)\sin\gamma(y - a)] \exp(i\alpha x)\, d\alpha$$

$$\xi_2(x, a - 0) = \int_{-\infty}^{+\infty} Z_2(\alpha) \exp(i\alpha x)\, d\alpha$$

$$\xi_2(x, -a + 0) = \int_{-\infty}^{+\infty} Z_2'(\alpha) \exp(i\alpha x)\, d\alpha$$

where $\gamma = \sqrt{\varkappa^2 - \alpha^2}$ is the branch of the root such that $\operatorname{Im}\gamma > 0$. These functions are obviously not independent, since

$$A(\alpha) = \frac{Z_2(\alpha)}{\sin 2\gamma a}, \quad B(\alpha) = -\frac{Z_2'(\alpha)}{\sin 2\gamma a}$$

Let us calculate the $y$-components of fluid velocity in regions 1 and 3, and introduce new unknown functions $V_1(\alpha)$ and $V_3(\alpha)$ using formulas

$$v_{1,3}(x, y) = \frac{\sigma^2}{\varkappa^2 h} \int_{-\infty}^{+\infty} V_{1,3}(\alpha) \exp[i\alpha x + i\gamma(\pm y - a)]\, d\alpha \tag{2.2}$$

The following relations are valid:

$$Z_{1,3}(\alpha) = \frac{V_{1,3}(\alpha)}{\pm\gamma - i\alpha l}$$

Applying formula (1.4) to function $\xi_2(x, y)$ and using the condition of continuity of velocities on the half-lines $|y| = a$, $x > 0$, we obtain for $Z_2(\alpha)$ and $Z_2'(\alpha)$ the following representations in terms of $V_1(\alpha)$ and $V_3(\alpha)$:

$$Z_2(\alpha) = i\frac{V_1(\alpha)(\alpha l \sin 2\gamma a - \gamma\cos 2\gamma a) + \gamma V_3(\alpha)}{(\gamma^2 + \alpha^2 l^2)\sin 2\gamma a}$$

$$Z_2'(\alpha) = i\frac{-\gamma V_1(\alpha) + (\gamma\cos 2\gamma a + \alpha l \sin 2\gamma a) V_3(\alpha)}{(\gamma^2 + \alpha^2 l^2)\sin 2\gamma a}$$

Substituting the integral representations of elevations into the last four of boundary conditions (1.3), adding and subtracting these, we obtain the following integral equations that are valid for $x > 0$:

$$\int_{-\infty}^{+\infty} \frac{\exp(i\alpha x)\, V_s(\alpha)\, L(\alpha)}{\alpha^2 - \eta^2\varkappa^2}\, d\alpha = \frac{a}{2\eta^2}\exp(i\eta\varkappa x) \tag{2.3}$$

$$\int_{-\infty}^{+\infty} \frac{\exp(i\alpha x)\, V_a(\alpha)\, M(\alpha)}{\alpha^2 - \eta^2\varkappa^2}\, d\alpha = -i\frac{a}{2\eta^2}\exp(i\eta\varkappa x)$$

$$V_a(\alpha) = (V_1(\alpha) - V_3(\alpha))/2, \quad V_s(\alpha) = (V_1(\alpha) + V_3(\alpha))/2$$

$$L(\alpha) = \frac{a\gamma\exp(-i\gamma a)}{\cos\gamma a}, \quad M(\alpha) = \frac{a\gamma\exp(-i\gamma a)}{\sin\gamma a}$$

Using the integral representation for velocities (2.2) and the first three of boundary conditions (1.3) we obtain the following integral equations that are valid for $x < 0$:

$$\int_{-\infty}^{+\infty} V_s(\alpha)\exp(i\alpha x)\, d\alpha = 0, \quad \int_{-\infty}^{+\infty} V_a(\alpha)\exp(i\alpha x)\, d\alpha = 0 \tag{2.4}$$

The input problem (1.1)—(1.5) has been, thus, reduced to solving the system of paired integral equations (2.3) and (2.4) for the unknown functions $V_s(\alpha)$ and $V_a(\alpha)$ of the complex variable $\alpha$.

To solve the system (2.3), (2.4) we factorize the kernels of integral equations $L(\alpha)$ and $M(\alpha)$ by representing these in the form $L(\alpha) = L_+(\alpha)L_-(\alpha)$ and $M(\alpha) = M_+(\alpha)M_-(\alpha)$, where the factors with the signs plus and minus are analytic in the upper and lower half-planes of the complex variable $\alpha$, respectively, and have no zeros there.

Since the factorization of kernels $L(\alpha)$ and $M(\alpha)$ had been described in numerous publications /3/, we present here only the final result of that procedure

$$\frac{1}{L_+(\alpha)}=\sqrt{\frac{\cos\varkappa a}{\varkappa a(1+\alpha/\varkappa)}}\exp\left[-i\frac{\pi}{4}+\frac{\alpha\gamma}{\pi}\ln\left(\frac{\alpha+i\gamma}{\varkappa}\right)+\right.$$
$$\left.\frac{i\alpha a}{\pi}\left(1-C+\ln\frac{\pi}{2\varkappa a}+i\frac{\pi}{2}\right)\right]\prod_{n=1}^{\infty}\left(1-\frac{\alpha}{\alpha_{2n-1}}\right)\exp\left[\frac{2i\alpha a}{\pi(2n-1)}\right]$$
$$\frac{1}{M_+(\alpha)}=\sqrt{\frac{\sin\varkappa a}{\varkappa a}}\exp\left[\frac{\alpha\gamma}{\pi}\ln\left(\frac{\alpha+i\gamma}{\varkappa}\right)+\frac{i\alpha a}{\pi}\left(1-C+\ln\frac{2\pi}{\varkappa a}+\right.\right.$$
$$\left.\left.i\frac{\pi}{2}\right)\right]\prod_{n=1}^{\infty}\left(1-\frac{\alpha}{\alpha_{2n}}\right)\exp\left[\frac{2i\alpha a}{2\pi n}\right]$$
$$\alpha_n=-\sqrt{\varkappa^2-\left(\frac{\pi n}{2a}\right)^2}$$

where $C$ is Euler's constant.

We seek a solution of system (2.3), (2.4) of the form

$$V_s(\alpha)=\frac{P}{L_-(\alpha)},\quad V_a(\alpha)=\frac{Q}{M_-(\alpha)} \tag{2.5}$$

where $P$ and $Q$ are unknown constants. With this selection of functions $V_s(\alpha)$ and $V_a(\alpha)$ Eqs. (2.4) are identically satisfied. For the determination of constants $P$ and $Q$ we substitute (2.5) into the integral equations (2.3) and, after the computation of residues in the pole $\alpha=\eta\varkappa$, obtain

$$P=\frac{\varkappa a}{2\pi i\eta L_+(\eta\varkappa)},\quad Q=\frac{-\varkappa a}{2\pi\eta M_+(\eta\varkappa)} \tag{2.6}$$

The derived solution (2.5), (2.6) satisfies condition (1.5) at the edge which in conformity with the theorem on the relation between the function asymptotics and its Fourier transform in the case of functions $V_s(\alpha)$ and $V_a(\alpha)$ assumes the form $V_s(\alpha), V_a(\alpha)\sim\alpha^{-1/2}$, as $\alpha\to\infty$. The determination of functions $V_s(\alpha)$ and $V_a(\alpha)$ together with functions

$$V_1(\alpha)=V_s(\alpha)+V_a(\alpha)=\frac{P}{L_-(\alpha)}+\frac{Q}{M_-(\alpha)} \tag{2.7}$$
$$V_3(\alpha)=V_s(\alpha)-V_a(\alpha)=\frac{P}{L_-(\alpha)}-\frac{Q}{M_-(\alpha)}$$

completely solves the problem of determination of fluid elevation in all regions of the tank.

## 3. Formulas for elevations.

We start with the determination of elevations in region 2 at $x<0$, i.e. inside the channel. Using (2.1) it is possible to obtain for elevations in that region the following integral representation:

$$\xi_2(x,y)=\int_{-\infty}^{+\infty}\frac{\exp(i\alpha x)}{\gamma^2+\alpha^2l^2}\left[\frac{V_s(\alpha)}{\cos\gamma a}(\gamma\sin\gamma y+\alpha l\cos\gamma y)+\right.$$
$$\left.\frac{V_a(\alpha)}{\sin\gamma a}(\alpha l\sin\gamma y-\gamma\cos\gamma y)\right]d\alpha \tag{3.1}$$

where the integral can be readily calculated using Jordan's lemma and obtain the residues of integrands at simple poles

$$\alpha_0=-\eta\varkappa,\quad \alpha_k=-\sqrt{\varkappa^2-\left(\frac{\pi k}{2a}\right)^2}\quad (k=1,2,\ldots)$$

As the result, we obtain

$$\xi_2(x,y)=-2\pi l\eta^2\frac{V_1(-\eta\varkappa)-V_3(-\eta\varkappa)\exp(-2l\eta\varkappa a)}{1-\exp(-4l\eta\varkappa a)}\times \tag{3.2}$$
$$\exp[-i\eta\varkappa x+l\eta\varkappa(y-a)]+$$
$$\sum_{k=1}^{\infty}[R_k\sin(\gamma_k y-\varphi_k)+T_k\cos(\gamma_k y-\varphi_k)]\exp(-i\alpha_k x)$$
$$\sin\varphi_k=\frac{\gamma_k}{\sqrt{\gamma_k^2+(\alpha_k l)^2}},\quad \cos\varphi_k=\frac{\alpha_k l}{\sqrt{\gamma_k^2+(\alpha_k l)^2}}$$
$$R_k=2\pi i(-1)^k\cos\frac{\pi k}{2}\frac{\gamma_k V_a(\alpha_k)}{\alpha_k a\sqrt{\gamma_k^2+(\alpha_k l)^2}},\quad \alpha_k=-\sqrt{\varkappa^2-\gamma_k^2}$$
$$T_k=2\pi i(-1)^k\sin\frac{\pi k}{2}\frac{\gamma_k V_s(\alpha_k)}{\alpha_k a\sqrt{\gamma_k^2+(\alpha_k l)^2}},\quad \gamma_k=\frac{\pi k}{2a}$$

The first term in (3.2) defines the Kelvin wave propagating in the channel and the infinite sum corresponds to progressing and damped waves, with the real $\alpha_k$ corresponding to progressing waves, and the imaginary one waves that are exponentially damped inside the channel with increasing distance from its open end. For a given dimensionless width $\varkappa a$ of the channel the number of progressing waves is equal to the integral part of the number $2\varkappa a / \pi$.

The following integral representation is valid for region 3:

$$\xi_3(x, y) = -\int_{-\infty}^{+\infty} \frac{\exp[i\alpha x - i\gamma(y + a)]}{\gamma + i\alpha l} V_3(\alpha)\, d\alpha \tag{3.3}$$

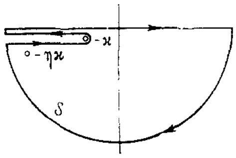


Fig.2

For $\varkappa < 0$ it is possible to use the theorem on residues (the integration path in the $\alpha$-plane is shown in Fig.2) and represent elevations $\xi_3(x, y)$ in the form

$$\xi_3(x, y) = -2\pi l\eta^2 V_3(-\eta\varkappa)\exp[-i\eta\varkappa x + l\eta\varkappa(y + a)] + \int_S \frac{\exp[i\alpha x - i\gamma(y + a)]}{\gamma + i\alpha l} V_3(\alpha)\, d\alpha \tag{3.4}$$

where the first term defines the Kelvin wave that propagates in region 3 in the negative direction of the $x$-axis along the wall $y = -a$, and the second term, the integral along the edges of slit $S$ (see Fig.2), defines the complex wave motion, which for any point of the considered region can be obtained by numerical integration on a computer.

Integral (3.3) for surface elevations at distances from the entry to the channel that are considerable compared with the wave length is estimated using the saddle-point method. For this we introduce polar coordinates $r, \theta$ using formulas

$$x = r\sin\theta, \quad y + a = -r\cos\theta, \quad |\theta| < \frac{\pi}{2}$$

The evaluation of the integral for $\xi_3(x, y)$ by the saddle-point method yields for the fluid elevation at considerable distance the following expression:

$$\xi_3(r, \theta) \sim \sqrt{\frac{2\pi}{\varkappa r}}\, \Theta(\theta) \exp\left[i\left(\varkappa r - \frac{\pi}{4}\right)\right] \tag{3.5}$$

$$\varkappa r \gg 1, \quad \Theta(\theta) = \frac{\cos\theta V_3(\varkappa\sin\theta)}{\cos\theta + il\sin\theta}$$

It can be shown that for $\varkappa r \gg 1$ the elevation in regions 1 and 3 is defined by a similar formula. Formula (3.5) shows that at large distances from the channel entry the elevations in the channel are of the form of divergent damped cylindrical waves with the angular distibution of amplitude $|\Theta(\theta)|$.

## 4. Propagation of the Kelvin wave from the channel.

Let a Kelvin wave of unit amplitude

$$\xi_0(x, y) = \exp[i\eta\varkappa x - l\eta\varkappa(y + a)] \tag{4.1}$$

propagate along the channel wall $y = -a$, $x < 0$ in the positive direction of the $x$-axis in the tank shown in Fig.1.

The problem of diffraction of such wave at the channel open end is solved by using the method expounded in Sects.1 and 2 which yields two systems of paired integral equations

$$\int_{-\infty}^{+\infty} V_a(\alpha)\exp(i\alpha x)\, d\alpha = 0, \quad x < 0 \tag{4.2}$$

$$\int_{-\infty}^{+\infty} \frac{V_a(\alpha) M(\alpha)\exp(i\alpha x)}{\alpha^2 - \eta^2\varkappa^2}\, d\alpha = ia\eta^2 \operatorname{sh}(l\eta\varkappa a)\exp(i\eta\varkappa x - l\eta\varkappa a), \quad x > 0$$

$$\int_{-\infty}^{+\infty} V_s(\alpha)\exp(i\alpha x)\, d\alpha = 0, \quad x < 0$$

$$\int_{-\infty}^{+\infty} \frac{V_s(\alpha) L(\alpha)\exp(i\alpha x)}{\alpha^2 - \eta^2\varkappa^2}\, d\alpha = \frac{a\eta^2}{2} \operatorname{ch}(l\eta\varkappa a)\exp(i\eta\varkappa x - l\eta\varkappa a), \quad x > 0$$

where the notation conforms to that in Sects.1 and 2. The factorization method is used for solving system (4.2) with the unknown functions assumed to be of the form

$$V_s(\alpha) = F / L_-(\alpha), \quad V_a(\alpha) = G / M_-(\alpha) \tag{4.3}$$

The constants $F$ and $G$ prove to be

$$F = \frac{\varkappa a}{2\pi i\eta L_+(\eta\varkappa)}[1 - \exp(-2l\eta\varkappa a)] \tag{4.4}$$

$$G = \frac{\varkappa a}{2\pi\eta M_+(\eta\varkappa)}[1 + \exp(-2l\eta\varkappa a)]$$

The obtained functions $V_s(\alpha)$ and $V_a(\alpha)$ completely solve the problem of diffraction of the Kelvin wave propagating from the channel into the tank.

The investigation of the fluid wave motions is carried out as described in Sect.3. Let us briefly define the field of wave elevations. A Kelvin wave reflected from the channel open end propagates in it in the negative direction of the $x$-axis, in which also propagate progressing waves (whose number is equal to the integral part of number $2\varkappa a / \pi$) and an infinite number of damped waves. Finally, the fluid elevations in all three regions at considerable distances from the channel inlet have the formof divergent damped cylindrical waves.Formulas (3.2),(3.4) and (3.5) determine the amplitudes of two Kelvins' progressive, damped and cylindrical waves. In this case the functions $V_1(\alpha)$ and $V_3(\alpha)$ in these formulas are calculated from formulas (2.7), taken into account (4.3) and (4.4).

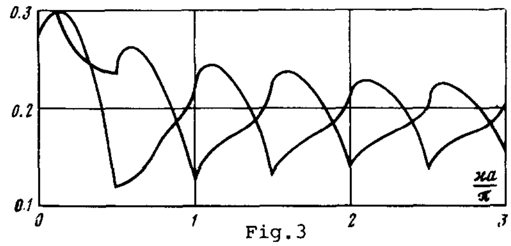

Fig.3

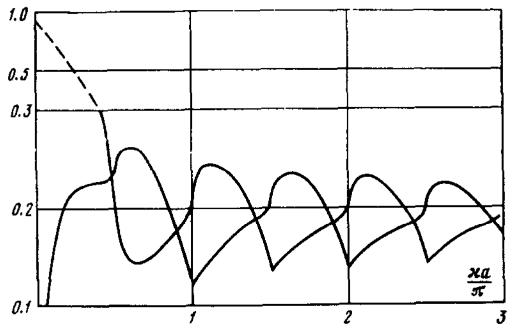

Fig.4

**5. Interpretation of results of numerical analysis.** The dependence of Kelvin wave amplitudes on the channel width $\varkappa a$ is shown in Fig.3 in the case when the initial wave propagates in region 1, while Fig.4 shows the same dependence in the case of the Kelvin wave issuing from the channel (the part of curve shown by the dash line is represented in a logarithmic scale). In each of these cases two Kelvin waves exist for any $\varkappa a$. It will be seen that for small $\varkappa a$ the Kelvin wave, after reaching the channel open end, is almost competely reflect from it, while in the first with the same $\varkappa a$ the amplitudes of both diffracted Kelvin waves are of comparable magnitude. As the "channel width" is increased for $\varkappa a = \pi n / 2\ (n = 1, 2, \ldots)$, progressing waves are generated. In Figs.3 and 4 the characteristic kinks at points $\varkappa a = \pi n / 2$ correspond to the emergence of progressing waves associated with the rearrangement of the amplitudes of the surface wave motions at the generation of a new progressing wave.

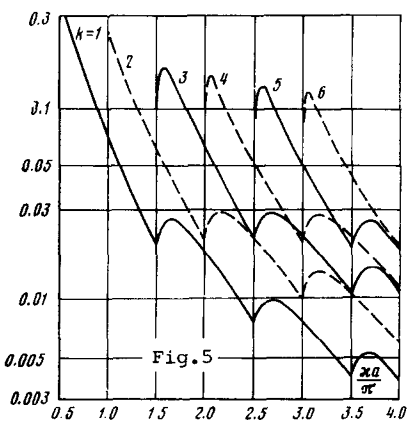

Fig.5

The described physical effect is characteristic not only for hydrodynamics but, also, for electrodynamics /4/ and nuclear physics /5/, and is called the threshold effect.

The dependence of the first progressing wave amplitudes on the channel width $\varkappa a$ is shown in Fig.5 in the case, when the initial wave propagates in region 1.
Since there are two independent sets of progressing waves in the channel, which differ in their symmetry properties, generation of a progressing wave of a particular symmetry leads to the rearrangement of the spectrum of waves of that symmetry only. This state can be examined in Fig.5.

Note that the results of investigation /6/ represent a particular case of the derived here solution.

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REFERENCES

1. SRETENSKII, L. N. The Theory of Wave Motions of Fluid. Moscow—Leningrad, ONTI, 1936.

2. NOBLE, B., Methods Based on the Wiener—Hopf Technique for the Solution of Partial Differential Equations, English translation, Pergamon Press, 1959.

3. MITTRA, R. and LI. S., Analytical Methods of the Theory of Wave Guides, Moscow, "Mir", 1974.

4. BOLOTOVSKII, B. M. and LEBEDEV, A. N. On threshold effects in classical electrodynamics. Zh. Eksper. i Teor. Fiz., Vol.53, No.4 (10), 1967.

5. BAZ', A. I., ZEL'DOVICH, Ia.B., and PERELOMOV, A. M., Scatter, Reactions and Disintegrations in Nonrelativistic Quantum Mechanics. Moscow, "Nauka", 1966.

6. GABOV, S. A., RUBAN, P. I. and SEKERZH—ZEN'KOVICH, S. Ia., Diffraction of Kelvin waves on a semi-infinite wall in a semibounded tank. English translation, J. U.S.S.R. Comput. Math. math. Phys. Pergamon Press, Vol.15, No.6, 1975.

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